

# Landau levels for electromagnetic wave

Vladimir A. Zyuzin<sup>1,2</sup>

<sup>1</sup>*Department of Physics and Astronomy and Nebraska Center for Materials and Nanoscience,  
University of Nebraska, Lincoln, Nebraska 68588, USA*

<sup>2</sup>*Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843-4242, USA*

In this paper we show that the frequencies of propagating electromagnetic wave (photon) in rotating dielectric media obey Landau quantization. We show that the degeneracy of right and left helicities of photons is broken on the lowest Landau level. In homogeneous space this level is shown to be helical, i.e. left and right helical photons counter-propagate.

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**Introduction.** Photons are spin-1 massless particles and are described by helicity, a scalar product of spin and propagation direction, which can only take +1 or -1 values. It was pointed out by M.V. Berry [1] that spin-1 boson can acquire a geometric phase, or the Berry phase. In other words, non-zero helicity of a photon is due to momentum-spin locking. Therefore when photon's momentum direction is adiabatically changed and returned to original direction, the spin acquires a phase - Berry phase. This phase was experimentally observed in a system of coiled optical fiber [2, 3]. Other interesting properties due to Berry phase of photon were studied in [4–6]. Due to non-trivial Berry phases of photons there exist chiral photon edge states [7–9] which are analogous to quantum Hall systems. Also an analog of topological insulator for photons can be drawn, see a problem to §68 of [10], [11], and [12].

Moreover, it is understood the motion of the media is an effective vector potential seen by the photons propagating in the media, see §57 of [10], and [13]. Using this concept, the Aharonov-Bohm effect for photons was proposed in [14] and is awaiting its experimental observation. In this paper we explicitly show that the equation describing propagation of photons inside a uniformly rotating dielectric media has a solution similar to the solution of the Schrödinger equation for electron in magnetic field, the Landau wave-functions and corresponding Landau levels [16]. Proposed in this paper Landau levels are another important ingredients in studies of the topological properties of photons. As an example, we find a gapless lowest Landau level, which for hypothetically infinite rotating system, becomes helical. Meaning that opposite photon helicities counter-propagate and do not mix with each other. This level is strikingly analogous to the chiral zeroth Landau level of three dimensional Dirac fermions, for example see [17]. In both cases the existence of such helical/chiral gapless level is due to non-trivial Berry curvature of a photon/Dirac fermion.

Recently there has been a relative interest in studying Landau levels for photons. For example, see an experimental work in which creation of synthetic Landau levels for photons in optical resonators was reported [18]. Another proposal of magnetic fluid of photons based on

non-linear effects has recently been put forward, see [19]. Present paper proposes a simple picture of creating Landau levels for photons in rotating media and hopefully it will inspire further research.

**Landau levels for electromagnetic waves.** Maxwell equations describing propagation of electro-magnetic wave in the dielectric media described by constant  $\epsilon$  and  $\mu$  in the absence of currents and charges are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad (1)$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \quad (2)$$

Assume that the dielectric media is moving with a speed  $\mathbf{v}$ . In the limit  $|\mathbf{v}|/c \ll 1$ , to the lowest order in  $|\mathbf{v}|/c$ , see §57 of [10], we write

$$\mathbf{D} = \epsilon \mathbf{E} + \frac{\epsilon\mu - 1}{c} [\mathbf{v} \times \mathbf{H}], \quad (3)$$

$$\mathbf{B} = \mu \mathbf{H} - \frac{\epsilon\mu - 1}{c} [\mathbf{v} \times \mathbf{E}]. \quad (4)$$

For the sake of generality we will assume an infinite system with homogeneous  $\epsilon > 0$  and  $\mu > 0$ . We then introduce two new fields as

$$\mathbf{F}^{(\pm)} = \sqrt{\epsilon} \mathbf{E} \pm i\sqrt{\mu} \mathbf{H}, \quad (5)$$

the  $\pm$  sign corresponds to photon helicity. For time-independent velocity  $\mathbf{v}$ , we then get the Maxwells equations as

$$\left( \nabla - \frac{\epsilon\mu - 1}{c^2} \mathbf{v} \partial_t \right) \times \mathbf{F}^{(\pm)} = \pm i \frac{\sqrt{\epsilon\mu}}{c} \partial_t \mathbf{F}^{(\pm)}, \quad (6)$$

$$\nabla \cdot \left[ \mathbf{F}^{(\pm)} \pm \frac{\epsilon\mu - 1}{i\sqrt{\epsilon\mu}c} [\mathbf{v} \times \mathbf{F}^{(\pm)}] \right] = 0, \quad (7)$$

which bear a similarity with Dirac equation, for a review of such approach see [15]. In this case the  $(\pm)$  play a role of helicity of photons. Here velocity  $\mathbf{v}$  plays a role of a vector potential of an effective magnetic field. We assume velocity to have a cylindrical symmetry, described by

$$\mathbf{v} = v(-y\mathbf{e}_x + x\mathbf{e}_y), \quad (8)$$

where  $v$  is angular velocity. Compare vector field (8) describing rotation with the symmetric gauge of the magnetic field. As an example, one can keep in mind a dielectric of cylindric form, which is rotating about its axis. However, as mentioned above, we are going to study a rotating system infinite in all directions. Even though it is not experimentally possible, we wish to study this case in order to understand the nature of solutions. For finite systems it is straightforward to set boundary conditions by integrating Eq. (6) and Eq. (7). We search for solutions in the form  $\propto e^{-i\omega t} e^{ip_z z}$ . For the sake of simplicity, introduce  $\Omega = \frac{\sqrt{\epsilon\mu}}{c}\omega$  and  $V = \frac{\epsilon\mu-1}{c^2}v\omega$ . In the following we choose  $V > 0$ , and as mentioned above assume a system to be infinite in all directions. Components of equation (6) are written as

$$\Pi_y F_z^{(\pm)} - ip_z F_y^{(\pm)} = \pm \Omega F_x^{(\pm)}, \quad (9)$$

$$ip_z F_x^{(\pm)} - \Pi_x F_z^{(\pm)} = \pm \Omega F_y^{(\pm)}, \quad (10)$$

$$\Pi_x F_y^{(\pm)} - \Pi_y F_x^{(\pm)} = \pm \Omega F_z^{(\pm)}, \quad (11)$$

where  $\Pi_y \equiv (-i\nabla_y + Vy)$  and  $\Pi_x \equiv (-i\nabla_x - Vy)$  is the updated momentum operator. After straightforward transformations, assuming all components of  $\mathbf{F}^{(\pm)}$  are non-zero, we obtain for  $F_z^{(\pm)}$  component an equation

$$(\Pi_y^2 + \Pi_x^2) F_z^{(\pm)} = (\Omega^2 - p_z^2) F_z^{(\pm)}. \quad (12)$$

The equation has exactly the form of Schrödinger equation describing an electron in a uniform magnetic field, chosen to be described in a symmetric gauge [16]. Hence we obtain the Landau solutions to the equation. We label the eigen values and energies by index  $n$ , and immediately write

$$F_{z,n}^{(\pm)} = e^{-V|\zeta|^2/2} \left( \partial_{\bar{\zeta}} - \frac{V}{2}\zeta \right)^n f(\bar{\zeta}), \quad (13)$$

$$\Omega_n^2 = 4V \left( n + \frac{1}{2} \right) + p_z^2, \quad (14)$$

where  $\zeta = x + iy$ ,  $\bar{\zeta} = x - iy$ , and  $f(\bar{\zeta})$  is an arbitrary function of  $\bar{\zeta}$ . The function can be presented through basis states as

$$f(\bar{\zeta}) = \sum_m f_m(\bar{\zeta}) = \sum_m \sqrt{N_m} \bar{\zeta}^m, \quad (15)$$

where  $N_m$  is a renormalization constant. Each  $f_m$  corresponds to  $m$ th orbit of the state on the  $n$ th level. For example, for  $n = 0$  each  $f_m$  corresponds to an orbit with a radius  $|\zeta|_m = \sqrt{m/V}$ .

Other two components of  $\mathbf{F}^{(\pm)}$  are expressed through  $F_z^{(\pm)}$  as

$$F_x^{(\pm)} = \frac{\pm \Omega}{\Omega^2 - p_z^2} \left( i\Pi_y \pm \frac{p_z}{\Omega} \Pi_x \right) F_z^{(\pm)}, \quad (16)$$

$$F_y^{(\pm)} = \frac{\pm \Omega}{i(\Omega^2 - p_z^2)} \left( \Pi_x \pm i\frac{p_z}{\Omega} \Pi_y \right) F_z^{(\pm)}. \quad (17)$$

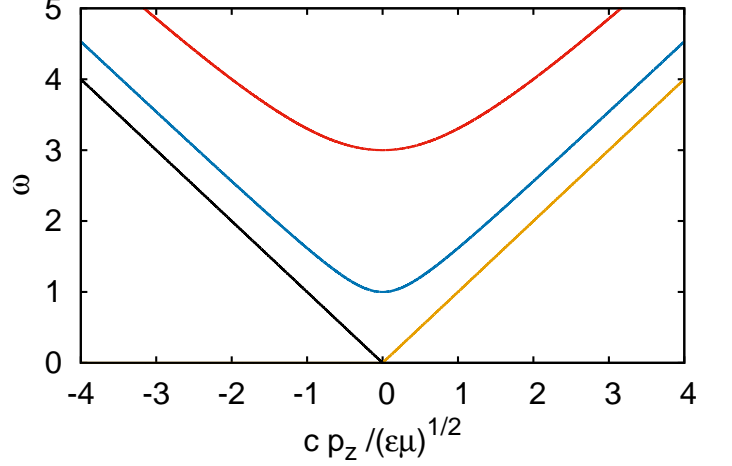


FIG. 1: The spectrum of the Landau levels for electromagnetic waves. Levels  $\omega_n$  described by 18 are plot in blue for  $n = 0$  and in red for  $n = 1$ . Two branches of the lowest Landau level described by Eq. (22) are plot in black and yellow. This level is gapples and helical, i.e. black corresponds to  $+$  helicity while yellow corresponds to  $-$  velocity for  $v > 0$  choice of angular velocity of rotation. Parameter  $2\frac{\epsilon\mu-1}{\epsilon\mu}v = 1$  in appropriate units of frequency is chosen for sake of simplicity.

The solutions found from equations (6) are consistent with equations (7). This can be seen by taking the divergence operation of the expression (6).

Obtained spectrum, keeping in mind that  $\omega > 0$  and  $\epsilon\mu - 1 > 0$ , is rewritten in a more transparent form

$$\omega_n = 2\frac{\epsilon\mu-1}{\epsilon\mu}v \left( n + \frac{1}{2} \right) + \sqrt{\left( 2\frac{\epsilon\mu-1}{\epsilon\mu}v \right)^2 \left( n + \frac{1}{2} \right)^2 + \frac{c^2 p_z^2}{\epsilon\mu}}. \quad (18)$$

Note that the spectrum does not depend on the helicity index,  $\pm$ . Hence the obtained solutions are degenerate in helicity. See Fig. [1] for schematical description of the Landau levels for  $n = 0$  and  $n = 1$ .

**Helical mode.** In the previous subsection we assumed that all components of  $\mathbf{F}^{(\pm)}$  are non-zero. We observe that there is an ambiguity of the expressions (16) and (17) if one sets  $F_z^{(\pm)} = 0$  and  $\Omega^2 = p_z^2$  in them. Hence, in the following we examine the  $F_z^{(\pm)} = 0$  case. In the following we will again consider infinite geometry and assume propagation in  $z$ -direction, such that  $\mathbf{F}^{(\pm)} \propto e^{-i\omega t} e^{ip_z z}$ . In this case we obtain for the equation (6),

$$ip_z F_y^{(\pm)} = \mp \Omega F_x^{(\pm)}, \quad (19)$$

$$ip_z F_x^{(\pm)} = \pm \Omega F_y^{(\pm)}, \quad (20)$$

$$\Pi_x F_y^{(\pm)} - \Pi_y F_x^{(\pm)} = 0, \quad (21)$$

from where one gets  $F_x^{(\pm)} = isF_y^{(\pm)}$ , with solution index  $s = \pm$  which does not correspond to helicity. For  $s = +$  we get  $(\Pi_x - i\Pi_y)F_y^{(\pm)} = -i(2\partial_{\bar{z}} + V\bar{z})F_y^{(\pm)} = 0$ , which can be met by  $F_y^{(+)} = f(\bar{z})\exp(-\frac{V}{2}|\bar{z}|^2)$  solution, with arbitrary function  $f(\bar{z})$ , given a requirement for the solution to decay as  $|\bar{z}| \rightarrow \infty$ . For  $s = -$  we get  $(\Pi_x + i\Pi_y)F_y^{(\pm)} = -i(2\partial_z - V\bar{z})F_y^{(\pm)} = 0$ , which can not be met by solutions decaying at large  $|\bar{z}|$ . Hence, only the  $s = +$  solution exists for  $v > 0$  choice of the angular velocity of dielectric media rotation. The spectrum of  $s = +$  solution for  $\pm$  helicity is

$$\omega = \mp \sqrt{\frac{c^2}{\epsilon\mu}} p_z, \quad (22)$$

together with the  $\omega > 0$  condition we obtain a requirement that  $+$  helical photons must have  $p_z < 0$ , and that  $-$  helical photons must have  $p_z > 0$ . Hence, such solution is purely helical, i.e. opposite helicities counter-propagate. See Fig. [1] for schematics. If the rotation is switched to an opposite,  $v \rightarrow -v$ , the propagation structure of different helicities switch places. Note that the spectrum (22) satisfies the  $\Omega^2 = p_z^2$  assumption which we started this subsection with. Hence solutions described by  $F_z^{(\pm)} \neq 0$  obtained in (13) and helical  $F_z^{(\pm)} = 0$  solution are different.

**Discussion.** For inhomogeneous case when  $\epsilon$  and  $\mu$  are functions of coordinates, the Eq. (6) and Eq. (7) will change, see for review [15]. For example imagine a rotating cylinder, in which case outside of the cylinder  $\epsilon = 1$  and  $\mu = 1$ , and inside  $\epsilon \neq 1$ . Helicities will then be mixed due to the inhomogeneous  $\epsilon$  and  $\mu$  functions. Helicity degeneracy of solutions of Maxwell equations described by (13) and (18) will not change in finite geometry. However, helical mode solutions (22) will change due to helicity mixing. Due to boundary conditions on the walls of rotating cylinder, it might be experimentally hard to observe the helical mode solutions. Also, it is important to note that there is a natural limit on the radius of rotating cylinder given by condition  $|\mathbf{v}|/c \ll 1$ . Therefore, if it will be possible to excite such helical mode in finite geometry, there will be spacial separation of co-propagating opposite helicity waves. For example in particular direction of cylinder rotation, intensity of  $+$  helicity of the  $p_z > 0$  wave will be peaked at the surface of the cylinder, while intensity of  $-$  helicity will be peaked closer to the axis of the cylinder.

We note a striking similarity of the obtained in the present paper helical mode to the chiral lowest Landau level of three dimensional Dirac fermion, for example see [17]. The similarity is due to non-trivial Berry curvature of photons and Dirac fermions.

From the structure of the Maxwell equation (12) we observe that effective charge of photon is its frequency, while the rotation plays a role of effective vector potential for the photon. Obtained conditions  $(\Pi_x - is\Pi_y)F_y^{(\pm)} =$

0, with  $s = \pm$  denoting relation  $F_x^{(\pm)} = isF_y^{(\pm)}$ , for the helical mode are helicity sensitive due to rotation. It can be checked that the solution becomes helicity degenerate when the rotation is switched off, as is expected for free space propagating photon.

Recently there were experiments on creating synthetic Landau levels for photons using resonators [18]. Various proposals on creating the angular momentum for photon [6] and [19] were put forward. The author believes that the obtained in the present paper results will excite further experimental interest in studies of Landau levels for photons.

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